2020

MATHEMATICS — HONOURS

Seventh Paper

(Module - XIII)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ respectively denote the set of natural numbers, integers, rational, real numbers and complex numbers.

Group - A

[Analysis - IV]

(Marks : 20)

Answer ***any one*** question.

1. (a) Let f and φ be two functions of x such that for some positive real number λ, $0 < f(x) \leqslant \lambda\varphi(x)$ for all $x \geqslant a$. If each of f and φ be integrable on $[a, X]$ for every $X > a$, prove that $\int_a^\infty f(x)dx$ converges if $\int_a^\infty \varphi(x)dx$ converges and $\int_a^\infty \varphi(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.

(b) Test the convergence of the integral $\int_0^1 \frac{\sqrt{x}}{e^{\sin x}-1}dx.$

(c) Establish the convergence of $\int_0^\infty \frac{x\log x}{\left(1+x^2\right)^2}dx$ and find its value. 8+6+6

2. (a) Show that $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}}dx$ is convergent.

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(b) State Abel's Test in connection with the convergence of improper integral of product of two functions over a bounded and closed interval. Using it, show that $\int_0^1 \frac{\log_e(1+x)\sin\frac{1}{x}}{x}\,dx$ is convergent.

(c) Express $\int_0^1 x^m\left(1-x^p\right)^n dx$ in terms of Beta function mentioning the conditions on m, n, p. Hence evaluate $\int_0^1 x^5\left(1-x^3\right)^{10} dx.$ 6+(2+4)+8

3. (a) Show that for $f(x)=\cos kx$ on $[-\pi, \pi]$, where k is not an integer,

$$\cos kx = \frac{\sin k\pi}{\pi}\left[\frac{1}{k} - \frac{2k\cos x}{k^2-1^2} + \frac{2k\cos 2x}{k^2-2^2} + \dots\right].$$

Deduce that $\pi\cos k\pi = \frac{1}{k} + 2k\sum_{n\in\mathbb{N}}\frac{1}{k^2-n^2}$.

(b) Evaluate $\int_0^1 dy\int_y^1 e^{x^2}\,dx$. 12+8

4. (a) Show that the integral $\iint_E e^{\frac{y-x}{y+x}}\,dx\,dy$, where E is the triangle with vertices at (0, 0), (0, 1) and (1, 0) is $\frac{1}{4}\left(e-\frac{1}{e}\right)$. 10

Or,

Evaluate $\iint_E x^{1/2}\,y^{1/3}(1-x-y)^{2/3}\,dx\,dy$, where E is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$. 10

(b) Show that the volume included between the elliptical paraboloid $2z=\frac{x^2}{p}+\frac{y^2}{q}$, the cylinder $x^2+y^2=a^2$ and the xy plane is $\frac{\pi a^4(p+q)}{8pq}$. 10

Or,

Let a function f be defined on a rectangle $R = [0, 1; 0, 1]$ as follows :

$$f(x, y) = \begin{cases} \frac{1}{2} & \text{when } y \text{ is rational} \\ x & \text{when } y \text{ is irrational} \end{cases}$$

Show that (i) $\int_0^1 dy \int_0^1 f(x, y)dx = \frac{1}{2}$ and (ii) $\int_0^1 dx \int_0^1 f(x, y)\,dy$ does not exist. 4+6

Group - B

[Metric space]

(Marks : 15)

5. Answer ***any one*** question :

(a) (i) For any two distinct points a, b in a metric space (X, d), prove that there exist disjoint open spheres with centres at a and b respectively.

(ii) In the metric space of real numbers $(\mathbb{R}, d)$ with the usual metric, let $\rho(A, B)$ be the distance between two subsets A, B of $\mathbb{R}$. Show that $\rho(A, B) = 0$ where $A = \mathbb{N}$ and $B = \left\{ n + \frac{1}{2n} : n \in \mathbb{N} \right\}$.

15

(b) (i) Let (X, d) be a metric space and $A, B \subset X$. Then show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ ($\overline{U}$ denote the closure of U).

(ii) If $\delta(A)$ and $\overline{A}$ denote diameter and closure of a set A in a metric space (X, d), then prove that $\delta(A) = \delta(\overline{A})$. 15

(c) (i) Consider the metric space $(\mathbb{R}^2, d)$ where $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ for $a = (0, 0) \in \mathbb{R}^2$ and any positive number r, describe the open ball $S(a, r)$ geometrically.

(ii) Let $C[a, b]$ be the set of all real valued continuous functions defined on $[a, b]$. Define $d(f, g) = \underset{f,g \in C[a,b]}{\text{Sup}} |f(t) - g(t)|$. Show that $A = \left\{ f \in C[a, b] : \inf_{x \in [a,b]} f(x) > 0 \right\}$ is an open set.

15

(d) Prove that $C[a, b]$, the set of all real valued continuous functions defined on $[a, b]$, is complete under the metric d where $d(f, g) = \sup\{|f(x) - g(x)| : a \le x \le b\}$ for all $f, g \in C[a, b]$. 15

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(e) (i) If a sequence $\{x_n\}_n$ is convergent in the metric space (X, d), then prove that for $a \in X$, the set $\{d(x_n, a) : n \in \mathbb{N}\}$ is bounded.

(ii) In the metric space $(\mathbb{R}, d)$ with usual metric, consider the sequence $\{F_n\}_n$ of sets where $F_n = \left[-5-\frac{1}{n}, -5\right] \cup \left[5, 5+\frac{1}{n}\right]$ for all $n \in \mathbb{N}$. Show that $\bigcap_{n=1}^{\infty} F_n$ is not singleton. 15

Group - C

[Complex Analysis]

(Marks : 15)

6. Answer ***any two*** questions :

(a) (i) Show that the image of a line T under the stereographic projection is a circle minus north pole in the Riemann sphere $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$.

(ii) Show that the function $\frac{\bar{z}}{z}$ is not continuous at the origin Z=0 for any choice of $f(0)$. 5+2½

(b) (i) If $f(z)$ and $\overline{f(z)}$ are both analytic in a region, then show that $f(z)$ is constant in that region.

(ii) Prove or disprove : If $f : S \to \mathbb{C}$ is differentiable on S, where $S \subseteq \mathbb{C}$ and $f'(z) = 0$ for all $z \in S'$, then f is a constant function on S. 5+2½

(c) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

Show that Cauchy-Riemann equations are satisfied at $z = 0$, but the derivative of f fails to exist there. 7½

(d) If $f(z)$ is an analytic function of $z = x + iy$, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\text{Re} f(z)|^2 = 2|f'(z)|^2$. 7½

(e) Use Milne-Thompson method to find an analytic function whose imaginary part is given by :

$$v(x, y) = 3x^2 y + y^3.$$ 7½